A Credit Cycle Model of Bank Loans and Corporate Debt: a Bank Capital View

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Abstract

In this paper, we present a continuous-time macro-finance framework, in which firms raise external funds by either issuing corporate debt or obtaining bank loans. Although banks are more efficient than debt holders when liquidating assets of firms with liquidity problems, the interest spread of bank loans must cover the intermediation cost and the risk premium. Because of the crucial feature that the risk premium depends on the financial health of the banking sector, the cost of obtaining bank loans endogenously fluctuates across business cycles. This continuous-time framework allows us to capture that although bank-financing is more cyclical and volatile than bond-financing in the long run, the rise in bond credit can make up the credit loss incurred by the drastic decline in loan supply during crises. Meanwhile, our model also captures the fact that costs of both bank and bond credit increase in recessions when bank capital deteriorates.

Keywords: bank capital, bank-financing, bond-financing, and credit cycles

1 Introduction

Bank loans and corporate debt are two most important financial instruments that firms in the real sector use to raise external funds. However, bank loans and corporate debt display very different behaviors in the business cycle. Based on the U.S. aggregate level data from 1953 to 2012, Becker and Ivashina (2014) observe that bank-financing is more volatile and cyclical than bond-financing and corporate debt is less affected by recessions than bank loans

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Nevertheless, when the economy is far away from its steady state such as the situation in the 2007-09 financial crisis, the credit loss caused by the shrinking of bank credit is made up by the surge of bond-financing to some degree, as Adrian et al. (2012) document for the 2007-09 financial crisis.

In this paper, we present a continuous-time macro-finance framework, in which bank capital plays a critical role. To the best of our knowledge, this is the first framework that captures both the heterogeneity of band-financing and bond-financing in the long run (documented in Becker and Ivashina (2014)) and the substitution of bond credit for bank credit in crises (highlighted in Adrian et al. (2012)). The key modelling feature that distinguishes our paper from many other papers on the same topic is that the cost of bank-financing depends on the financial health of the intermediary sector and thus fluctuates endogenously across business cycles.

In our framework, firms choose to either issue corporate debt directly or receive loans from banks. Firms differ in the likelihood that they will have liquidity problems. Although the liquidity problem may or may not result in fundamental losses to a firm, creditors of the firm would like to liquidate its assets to protect their investments from firms’ opportunistic behaviors. On the one hand, banks are more efficient than debt holders in terms of liquidating firms’ assets (Bolton and Freixas, 2000); on the other hand, firms need to compensate banks for intermediation costs of bank loans in addition to the risk premium for the aggregate risk that banks are exposed to. Therefore, it is not difficult to see that firms with relatively low liquidity risks tend to choose corporate debt rather than bank loans since the likelihood that these firms would have to face costly liquidation is low. This is consistent with empirical findings in Rauh and Sufi (2010).

When the banking sector is well capitalized, it channels more funds from creditors to firms. This in turn improves the aggregate productivity of the economy and boosts asset prices. However, when an adverse aggregate shock hits the economy, bank capital absorbs a disproportionately large share of the shock due to the use of leverage. As a result, the supply of bank loans shrinks, the aggregate productivity deteriorates, and asset prices decline. The depreciation of asset prices in turn hurts banks’ balance sheets and lowers the supply of bank loans further. Therefore, the financial intermediary sector amplifies the effect of the initial aggregate shock. We name the impact of this amplification on asset prices as endogenous risk.

The share of bank capital in total wealth is a key endogenous state variable that drives the credit cycle of the economy. When the share of bank capital is thin in the economy, the supply of bank loans are relatively small and thus the interest rate on bank loans is high. Therefore, when a negative shock hits the economy, the marginal value of bank capital
increases because the interest rate on bank loans goes up. The change in the marginal of bank capital caused by the aggregate shock effectively determines the risk premium that banks requests for each unit of credit banks lend out. The cost of raising bank credit endogenously fluctuates because it depends on the risk premium, which in turn relies on the financial health of the banking sector. In economic booms when the banking sector is financially sound, the cost of bank-financing is relatively low.

The result that the fluctuation in the cost of bank-financing is endogenous is in contrast with many other related papers such as De Fiore and Uhlig (2011), De Fiore and Uhlig (2015), and Crouzet (2014). Since these papers model the surge in the cost of bank financing as an exogenous shock, they could not have rich characterizations of dynamics of bank-financing and bond-financing as what we do in our paper. One exception is Rampini and Viswanathan (2015), which also endogenize the cost of financial intermediation. But, this paper does not address the substitution between bank credit and bond credit.

Across credit cycles, firms with modest and high liquidity risks tend to choose bank-financing when the banking sector is well capitalized and bank loans are relatively less expensive. However, firms with low liquidity risks always stick to bond-financing because high liquidation cost associated with bond-financing has little effect on their borrowing costs.

Bank-financing is pro-cyclical in our model. During economic upturns when the financial condition of the banking sector improves, it is relatively cheap to raise bank credit. Thus, more firms choose bank-financing, and these firms also take high leverage because of low endogenous risks in addition to cheap bank credit.

Bond-financing is less volatile than bank-financing as a result of two opposing effects. At the extensive margin, less firms choose to issue corporate debts during economic booms when bank loans are relatively cheap. Nevertheless, at the intensive margin, firm that still raise bond credit would like to issue more corporate debt because of low endogenous risks in economic booms. In financial crises, however, more firms issue corporate debt because of the rising cost of bank loans and firms also take high leverage due to high returns of holding assets caused by low asset prices. Therefore, we observe that the rise in bond credit in crises can make up the loss caused by the decline in the supply of bank loans.

The price impact of the substitution of bond financing for bank financing is significant. As bank loans become more expensive, less firms are able to raise external credit and those who switch to bond financing are still subject to high liquidation costs. As a consequence, asset prices have to drop substantially so that firms who still can raise external funds are willing to take high leverage and the aggregate productivity does not decline much. Endogenous risks increase significantly because of this mechanism. In addition, the result of our model is in line with the fact that the borrowing costs of both bank credit and bond credit increase
in recessions when the banking sector is financially unsound.

The structure of the rest of the paper follows. Section 2 describes the set-up of the model and defines the equilibrium. In Section 3, we characterize the optimal choice of individual agents and the Markov equilibrium that this paper focuses on. Section 4 illustrates key properties of the Markov equilibrium with numerical examples. Lastly, Section 5 concludes the paper.

2 Model

In this section, we build a macro-finance model, in which firms can either directly issue corporate debt or raise credit via financial intermediaries. The economy is infinite-horizon, continuous-time, and has two types of goods: perishable final goods and durable physical capital goods. Final goods serve as the numéraire.

Two groups of agents populate in the economy: households and bankers. Households have logarithmic preferences and bankers are risk-neutral. Both households and bankers have time discount factor $\rho$. Neither group accepts negative consumption. Households hold physical capital goods and produce final goods, and bankers specialize in financial intermediation.

In the beginning of each period, each household becomes a high productive expert at probability $\alpha$. The event that a household turns into an expert is independent across all households and different periods. Hereafter, we refer to households who do not become experts as normal households.

2.1 Technology

In each period, an expert can produce $ak_t$ units of final goods with $k_t$ units of physical capital. Normal households, who are less productive, also have a linear production function $y_t = a_hk_t$, where $a_h < a$. Both experts and households can convert $\iota_t k_t$ units of final goods into $k_t \Phi(\iota_t)$ units of physical capital, where

$$\Phi(\iota_t) = \log(\iota_t \phi + 1) / \phi.$$ 

Thus, there is technological illiquidity on the production side. Physical capital in the possession of experts depreciates at rate $\delta$ and, in normal households’ hands, physical capital depreciates at rate $\delta_h$.

Exogenous aggregate shocks are driven by a standard Brownian motion $\{Z_t, t \geq 0\}$. In the absence of any idiosyncratic shock, physical capital managed by an expert evolves
according to
\[ dk_t = (\Phi(t_t) - \delta)k_t dt + \sigma k_t dZ_t. \]

Similarly, physical capital managed by normal households follows
\[ dk_t = (\Phi(t_t) - \delta_h)k_t dt + \sigma k_t dZ_t. \]

When a household becomes an expert, he also receives a random draw \( \lambda \) from a distribution \( G[\lambda_{\text{min}}, \lambda_{\text{max}}] \), where \( 0 \leq \lambda_{\text{min}} < \lambda_{\text{max}} \leq 1 \). Suppose an expert with a random draw \( \lambda \) establishes several firms in a period. Each of these firms may experience a liquidity problem at probability \( \lambda \) in the period. Whether a firm has a liquidity problem is independent across all firms managed by the same expert. Thus, an expert establishes an infinitely number of firms to diversify the idiosyncratic liquidity risk. We will illustrate the consequence of a liquidity shock for a firm in the following section since the consequence depends on the financing method of the firm. Normal households are not subject to any idiosyncratic shock.

2.2 Corporate Debt, Bank Loan, and Liquidation

A firm managed by an expert can raise credit either from issuing corporate debt or from a bank. In addition, we assume that no firm can issue outside equity and that all firms have limited liability.

In the corporate debt market there is one and only one passive mutual fund, which pools households’ savings and invests in corporate debt. If a firm that issues corporate debt receives the liquidity shock, the mutual fund seizes a part of the firm’s physical capital in proportion to the firm’s debt-to-asset ratio and liquidate these assets.\(^1\) In the process of liquidation, the depreciation rate of physical capital rises to \( \kappa_d + \delta \). The mutual fund is also exposed to the aggregate risk because it holds physical capital in the event of liquidity shocks.

For simplicity, we assume that the borrowing rate that the mutual fund charges is the risk-free rate plus the expected loss due to costly liquidation, and that the mutual fund promises the risk-free rate \( r_t \) to its investors. Any loss or profit that the mutual fund has is shared by all households (including experts) in proportion to their net worth.

\(^1\) The micro-foundation for the mutual fund’s liquidation choice is the following. Suppose a firm raise \( L \) units of capital from the mutual fund and put down physical capital worth of \( L \) as collateral. If the firm has a liquidity problem, it may have a fundamental problem at a rate that is greater than \( \kappa_d \) and the fundamental problem will destroy the firm’s physical capital completely. Therefore, the mutual can avoid the fundamental problem by liquidating the underlying asset of the collateralized borrowing. In the model, we omit the fundamental problem setting and directly assume that the mutual fund will liquidate assets that back its lending.
If a firm raises credit from a bank, the liquidation process is the same except that the depreciation rate of physical capital only increases to $\kappa + \delta$, where $\kappa < \kappa^d$. However, bank lending involves a intermediation cost $\tau$ for each dollar lent to a firm. Since banks cannot share their exposure to the aggregate risk with households, banks will ask for a risk premium and this risk premium will depends on the liquidity risk of a firm. Thus, banks charge firms with different $\lambda_t$ for different borrowing rate $r^\lambda_t$. Similar to the mutual fund, banks raise funds from households and promise the risk-free rate $r_t$.

No liquidation process is involved if a firm is self-financed.

### 2.3 A Household’s Problem

We conjecture that the equilibrium price of physical capital follows

$$dq_t = \mu^q_t q_t dt + \sigma^q_t q_t dZ_t,$$

then the rate of return from holding physical capital for an expert in the absence of any shock is

$$R_t dt \equiv \left(\frac{a - \nu_t}{q_t} + \Phi(\nu_t) - \delta + \mu^q_t + \sigma^q_t\right) dt.$$  

The corresponding term for a normal household is

$$R^h_t dt \equiv \left(\frac{a_h - \nu_t}{q_t} + \Phi(\nu_t) - \delta_h + \mu^q_t + \sigma^q_t\right) dt.$$  

Thus, a normal household’s dynamic budget constraint is

$$\frac{dw^h_t}{w^h_t} = x_t(R^h_t dt + (\sigma + \sigma^q_t) dZ_t) + (1 - x_t)r_t dt - c_t dt,  \quad (2)$$

where $x_t$ is the portfolio weight on physical capital. Without loss of generality, we drop the loss or benefit that the normal household takes from the mutual fund.

If a household becomes an expert and obtains the random draw $\lambda_t$, then the expert will choose the financing method for his firms: corporate debt, bank loans, or self-financing. Since all of the expert’s firms are identical prior to the realization of the liquidity shock, financing decisions of all firms managed by the expert are the same. Thus, the debt-to-equity ratio of these firms is also the same, which is exactly the expert’s debt-to-net-worth ratio.

If the expert issues corporate debt, the law of motion for his net worth is

$$\frac{dw_t}{w_t} = R_t dt + (\sigma + \sigma^q_t) dZ_t + b_t \left( (R_t - \lambda_t \kappa^d - r_t) dt + (1 - \lambda_t)(\sigma + \sigma^q_t) dZ_t \right),  \quad (3)$$
where $b_t (\geq 0)$ is firms’ debt-to-equity ratio. By the Law of Large Numbers, the liquidity risk implies that the mutual fund seizes $\lambda_t$ proportion of the expert’s physical capital that is funded by corporate debt. As a result, the expert partially unloads his exposure to the aggregate risk, $\lambda_t (\sigma + \sigma_q^t) dZ_t$. In addition to the risk-free rate, the mutual fund will charge a premium that covers the loss caused by the additional depreciation of the liquidation process, $\lambda_t \kappa d_t$. Similarly, if the expert raises credit from a bank, his net worth evolves according to

$$
\frac{dw_t}{w_t} = R_t dt + (\sigma + \sigma_t^q) dZ_t + l_t \left( (R_t - \lambda_t \kappa - r_t^\lambda) dt + (1 - \lambda_t) (\sigma + \sigma_t^q) dZ_t \right),
$$

(4)

where $l_t (\geq 0)$ is firms’ loan-to-equity.

If the expert finances his investments with internal funds only, the law of motion for his net worth is

$$
\frac{dw_t}{w_t} = R_t dt + (\sigma + \sigma_t^q) dZ_t + b_t \left( (R_t - r_t) dt + (\sigma + \sigma_t^q) dZ_t \right),
$$

(5)

where $-1 \leq b_t \leq 0$.

Taking $\{q_t, r_t, r_t^\lambda, \lambda \in [\lambda_{\min}, \lambda_{\max}], t \geq 0\}$ as given, a household chooses $\{c_t, b_t, l_t, t \geq 0\}$ to maximize his life-time expected utility

$$
E_0 \left[ \int_0^\infty e^{-\rho t} \ln(c_t) \, dt \right],
$$

(6)
given that his net worth evolves in each period according to either of equation (2) – (5) depending on exogenous shocks and the expert’s own financing decision in the corresponding period.

### 2.4 A Banker’s Problem

The risk of a bank loan only depends on the liquidity risk of the borrower. Thus, a bank charges an expert with random draw $\lambda_t$ for borrowing rate $r_t^\lambda$. Hereafter, we refer the type of a bank loan to the liquidity risk of the borrower $\lambda$. Let $\int_{\lambda_{\min}}^{\lambda_{\max}} x_t^\lambda d\lambda$ denotes the ratio of the bank’s total lending to its capital, where $x_t^\lambda$ denotes the density of the ratio with respect to loans of type $\lambda$. Hence, a banker’s net worth $n_t$ evolves according to

$$
\frac{dn_t}{n_t} = \int_{\lambda_{\min}}^{\lambda_{\max}} x_t^\lambda \left( r_t^\lambda dt + \lambda (\sigma + \sigma_t^q) dZ_t - \tau \right) d\lambda + \left( 1 - \int_{\lambda_{\min}}^{\lambda_{\max}} x_t^\lambda d\lambda \right) r_t dt - \frac{dC_t}{n_t},
$$

(7)
where \( C_t \) denotes the cumulative consumption flows up to period \( t \). The banker is exposed to the aggregate risk \( x_t^\lambda \lambda (\sigma + \sigma^2_t) dZ_t \) because she takes over and resell the physical capital that backs her lending. Taking \( \{ q_t, r_t, r_t^\lambda, \lambda \in [\lambda_{\min}, \lambda_{\max}], t \geq 0 \} \) as given, a banker chooses \( \{ C_t, x_t^\lambda, \lambda \in [\lambda_{\min}, \lambda_{\max}], t \geq 0 \} \) to maximize her life-time expected utility

\[
E_0 \left[ \int_0^\infty e^{-\rho t} dC_t \right]
\]

subject to the dynamic budget constraint (7).

### 2.5 Equilibrium

The aggregate shock \( \{ Z_t, t \geq 0 \} \) drives the evolution of the economy. \( I = [0, 1] \) denotes the set of all households and \( J = (1, 2) \) the set of bankers. After the productivity shock is revealed in period \( t \), \( I_t^e \) denotes the set of experts in period \( t \), \( I_t^{\lambda} \) the set of experts with liquidity risk \( \lambda \) in period \( t \), and \( I_t^h \) the set of normal households in period \( t \).

**Definition 1** Given the initial endowments of physical capital \( \{ k^i_0, k^j_0; i \in I, j \in J \} \) to households and bankers such that

\[
\int_0^1 k^i_0 di + \int_1^2 k^j_0 dj = K_0,
\]

an equilibrium is defined by a set of stochastic processes adapted to the filtration generated by \( \{ Z_t \}_{t=0}^\infty \): the price of physical capital \( \{ q_t \}_{t=0}^\infty \), risk-free rate \( \{ r_t \}_{t=0}^\infty \), the interest rate of bank loan \( \{ r_t^\lambda, \lambda_{\min} \leq \lambda \leq \lambda_{\max} \}_{t=0}^\infty \), wealth \( \{ W_t^i, N_t^j, i \in I, j \in J \}_{t=0}^\infty \), investment decisions \( \{ i^t_i, i \in I \}_{t=0}^\infty \), asset holding decisions \( \{ x_t^i, i \in I_t^e \}_{t=0}^\infty \) of normal households, corporate debt financing decisions \( \{ b_t^i, i \in I_t^e \}_{t=0}^\infty \) of experts, bank financing decisions \( \{ l_t^i, i \in I_t^h \}_{t=0}^\infty \) of experts, bank lending \( \{ x_t^{\lambda, j}, j \in J \}_{t=0}^\infty \) and consumption \( \{ c_t^i, c_t^j, i \in I, j \in J \}_{t=0}^\infty \) such that

1. \( W_0^i = k_0^i q_0 \) and \( N_0^j = k_0^j q_0 \) for \( i \in I \) and \( j \in J \);
2. each household and each banker solve for their problems given prices;
3. markets for final goods and physical capital clear, that is,

\[
\int_0^1 c_i^t d i + \int_1^2 d C_t^j d j + \frac{1}{q_t} \int_1^2 \int_{\lambda_{\min}}^{\lambda_{\max}} \tau n_t^{\lambda, j} x^\lambda x^\lambda d \lambda d j = \frac{1}{q_t} \int_{i \in I_t^e} (a - i_t^i) w_t^i x_t^i d i +
\]

\[
\frac{1}{q_t} \int_{i \in I_t^e} (a - i_t^i) w_t^i (1 + b_t^i) d i + \frac{1}{q_t} \int_{i \in I_t^e} (a - i_t^i) w_t^i (1 + l_t^i) d i
\]

for the market of final goods, and

\[
\frac{1}{q_t} \int_{i \in I_t^e} w_t^i (1 + b_t^i) d i + \frac{1}{q_t} \int_{i \in I_t^e} w_t^i (1 + l_t^i) d i + \frac{1}{q_t} \int_{i \in I_t^h} w_t^i x_t^i d i = K_t
\]
for the market of physical capital goods, where \( K_t \) evolves according to

\[
\begin{align*}
\text{d}K_t &= \frac{1}{q_t} \int_{i \in I_t} (\Phi(i'_t) - \delta^h) w_t^i x_t^i \text{d}i + \frac{1}{q_t} \int_{i \in I_t} (\Phi(i'_t) - \delta) w_t^i (1 + b_t^i) - 1_{b_t^i > 0} \lambda \kappa w_t^ib_t^i \text{d}i \\
&+ \frac{1}{q_t} \int_{i \in I_t} (\Phi(i'_t) - \delta) w_t^i (1 + l_t^i) - \lambda \kappa w_t^i l_t^i \text{d}i
\end{align*}
\]

4. the bank loan market for each type of experts clears, that is, for \( \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \)

\[
\int_{i \in I_t^{\lambda}} w_t^i l_t^i \text{d}i = \int_{1}^{2} \int_{1}^{2} n_t^j x_t^\lambda \text{d}j.
\]

The credit market for corporate debt clears automatically by Walras’ Law.

3 Solving for the Equilibrium

Bank capital is important for the equilibrium, especially when productive experts are not sufficiently wealthy to hold all physical capital in the economy. We expect that the price of physical capital declines as the share of banking capital shrinks due to adverse exogenous shocks.

To solve for the equilibrium, we first derive first-order conditions with respect to optimal decisions of both households and bankers; secondly, we solve for the law of motion for the endogenous state variable, the share of banking capital in the economy’s total wealth, based on market clearing conditions as well as first-order conditions; lastly, we use first-order conditions and the state variable’s law of motion to define differential equations that endogenous variables such as the price of physical capital satisfy.

3.1 Households’ Optimal Choices

Households have logarithmic preferences. In the following discussion, we will take advantage of three well-known properties with respect to logarithmic preferences in the continuous-time setting: 1) a household’s consumption \( c_t \) is \( \rho \) proportion of her wealth \( w_t \) in the same period, i.e.,

\[
c_t = \rho w_t;
\]

2) a household’s portfolio weight on a risky investment is such that the Sharpe ratio of the risky investment equals the percentage volatility of her wealth; 3) a household’s life-time
expected utility, i.e., continuation value can be expressed as \( \ln(w_t)/\rho + h_t \), where \( h_t \) is not directly affected by the household’s decisions.

A household may turn into a normal household or an expert in the beginning of each period. Nevertheless, regardless of whether a household becomes an expert or not, his investment rate \( \iota_t \) always maximizes \( \Phi(\iota_t) - \iota_t/q_t \). The first-order condition implies that

\[ \Phi'(\iota_t) = \frac{1}{q_t}. \]  

(10)

Since whether a household becomes an expert or not is exogenous, we can characterize optimal portfolio choices of a normal household and optimal decisions of an expert separately. Given the second property discussed above, it is straightforward to derive a normal household’s optimal portfolio weight on the physical capital \( x_t \), which satisfies

\[ x_t = \max\{R^b_t - r_t, 0\} \frac{1}{(\sigma + \sigma^q)^2}. \]  

(11)

However, it is more complicated to characterize an expert’s portfolio choice because he needs to pick the financing method among three candidates: corporate debt, bank loans, and self-financing.

### 3.1.1 Experts’ Portfolio Choices

Intuitively, an expert chooses his financing method and portfolio weight to maximize the expected growth rate of his continuation value

\[ \frac{1}{\rho} E[\frac{d\ln(w_t)}{dt}] = \frac{1}{\rho} \left( \mu^w_t - \frac{1}{2}(\sigma^w)^2 \right), \]

where \( \mu^w_t \) is the percentage drift of the household’s wealth and \( \sigma^w \) the percentage volatility.\(^2\)

Before analysing an expert’s financing decision, we characterize his portfolio choice given the financing method and his random draw \( \lambda_t \). Suppose that the expert decides to issue corporate debt, the optimal debt-to-equity \( b_t \) of his firms solves

\[ G^b_t \equiv \max_{b \geq 0} \left\{ R_t + b(R_t - \lambda_t \kappa^d - r_t) - 0.5(1 + b(1 - \lambda_t))^2 (\sigma + \sigma^q)^2 \right\}. \]

\(^2\)Formal discussions of experts’ optimal portfolio choices are in the appendix.
Thus,

\[ b_t = \begin{cases} 
0, & \text{if } R_t - \lambda_t \kappa d - r_t - (1 - \lambda_t)(\sigma + \sigma_t^q)^2 < 0 \\
\frac{R_t - \lambda_t \kappa d - r_t - (1 - \lambda_t)(\sigma + \sigma_t^q)^2}{(1 - \lambda_t)^2(\sigma + \sigma_t^q)^2}, & \text{otherwise}
\end{cases} \]  

(12)

Secondly, if the expert chooses bank loans, then the optimal debt-to-equity ratio of his firms \( l_t \) maximizes

\[ G^d_t = \max_{l \geq 0} \left\{ R_t + l(R_t - \lambda_t \kappa - r_t^\lambda) - 0.5(1 + l(1 - \lambda_t))(\sigma + \sigma_t^q)^2 \right\}, \]

and

\[ l_t = \begin{cases} 
0, & \text{if } R_t - \lambda_t \kappa - r_t^\lambda - (1 - \lambda_t)(\sigma + \sigma_t^q)^2 < 0 \\
\frac{R_t - \lambda_t \kappa - r_t^\lambda - (1 - \lambda_t)(\sigma + \sigma_t^q)^2}{(1 - \lambda_t)^2(\sigma + \sigma_t^q)^2}, & \text{otherwise}
\end{cases} \]  

(13)

Compared with equation (12), equation (13) indicates that when the interest rate on bank loans \( r_t^\lambda \) is close to the risk-free rate \( r_t \) firms tend to choose bank loans since they can obtain relatively high leverage.

Lastly, if the expert only resorts to internal funds, then the portfolio weight on physical capital \( 1 + b_t \) solves

\[ G^i_t = \max_{-1 \leq b \leq 0} \left\{ R_t + b(R_t - r_t) - 0.5(1 + b)^2(\sigma + \sigma_t^q)^2 \right\}, \]

and

\[ b_t = \begin{cases} 
0, & \text{if } R_t - r_t - (\sigma + \sigma_t^q)^2 > 0 \\
-1, & \text{if } R_t - r_t < 0 \\
\frac{R_t - r_t - (\sigma + \sigma_t^q)^2}{(\sigma + \sigma_t^q)^2}, & \text{otherwise}
\end{cases} \]  

(14)

In summary, the expert’s optimal financing choice and portfolio choice solve \( \max \{ G^b_t, G^d_t, G^i_t \} \).

Bank loans are at least as expensive as corporate debt \( r_t^\lambda \geq r_t \). Experts with sufficiently low liquidity risks \( \lambda_t \) prefer issuing corporate debt because of the relatively small liquidation cost \( \lambda_t \kappa d \). However, experts suffering from high liquidity risks are willing to accept costly bank loans in exchange for relatively efficient liquidation. When bank loans are very expensive, experts with high liquidity risks may refrain themselves from raising outside credit to avoid the costly liquidation process.

### 3.2 Banker’s Optimal Choices

The marginal value of bank capital in period \( t \) depends on the future investment opportunity that banks have, i.e., the stochastic process \( \{ r_t^\lambda, s \geq t \} \). The return of bank lending, in turn,
depends on the path of the banking sector’s total capital \( N_t \): ample bank capital implies descending borrowing costs of bank loans. As in Brunnermeier and Sannikov (2014), we let \( \theta_t \) denote the marginal value of bank capital. Thus, the life-time expected utility of a banker is

\[
\theta_t n_t \equiv E_t \left[ \int_t^{\infty} e^{-\rho(s-t)} dC_s \right].
\]

Lemma 1 in Brunnermeier and Sannikov (2014) suggests that if the maximum life-time expected utility of a banker is finite under stochastic processes of the interest rate on bank loans and the risk-free rate \( \{r^\lambda_t, r_t, t \geq 0\} \), we can characterize the process \( \{\theta_t, t \geq 0\} \) and the banker’s portfolio choice by solving the Hamilton-Jacob-Bellman equation with respect to \( \theta_t n_t \),

\[
\rho \theta_t n_t dt = \max_{dC_t \geq 0, x^\lambda_t \geq 0} \left[ n_t dC_t + E[\theta_t n_t] \right],
\]

where

\[
E[\theta_t n_t] = n_t \mu_t^\theta + n_t \int_{\lambda_{\min}}^{\lambda_{\max}} (x^\lambda_t r^\lambda_t - x^\lambda_t \tau + (1 - x^\lambda_t) r_t) d\lambda - dC_t
\]

\[
+ n_t \sigma_t^\theta \int_{\lambda_{\min}}^{\lambda_{\max}} x^\lambda_t \lambda (\sigma + \sigma_t^\theta) d\lambda.
\]

The following proposition summarizes the characterization of \( \theta_t, t \geq 0 \) and the banker’s portfolio choice.

**Proposition 1** Suppose the law of motion for the stochastic process \( \{\theta_t, t \geq 0\} \) is denoted by

\[
\frac{d\theta_t}{\theta_t} = \mu_t^\theta dt + \sigma_t^\theta dZ_t,
\]

and \( \theta_t n_t \) is finite given \( \{r^\lambda_t, r_t, r \geq 0\} \), then \( \{\theta_t, x^\lambda_t, dC_t, t \geq 0\} \) is such that

1. \( \theta_t \geq 1 \) for \( t \geq 0 \); \( dC_t > 0 \) if \( \theta_t = 1 \);
2. \( \rho = \mu_t^\theta + r_t \)
3. for \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \),

\[
\begin{cases}
  x^\lambda_t > 0, & \text{if } r^\lambda_t - \tau - r_t = -\sigma_t^\theta \lambda (\sigma + \sigma_t^\theta) \\
  x^\lambda_t = 0, & \text{if } r^\lambda_t - \tau - r_t < -\sigma_t^\theta \lambda (\sigma + \sigma_t^\theta)
\end{cases}
\]

4. the transversality condition \( E[e^{-\rho t}\theta_t n_t] \to 0 \) holds.

Banks pay out dividends and bankers consumes only if the marginal value of bank capital equals 1; otherwise, banks retain their earnings since it is the only option for them to
accumulate bank capital. The interest rate on bank loans $r^λ_t$ relies on the intermediation cost $τ$, banks’ exposure to aggregate risk $λ(σ + σ^f_t)$, and banks’ risk appetite $1/(-σ^θ_t)$. The borrowing cost of bank loans for firms fluctuates endogenously not just because the price volatility of physical capital changes over time but also because banks’ risk appetite varies across business cycles.

### 3.3 Market Clearing

Equation (17) implies that the supply of bank loans is indeterminate given that the interest rate on bank loans equals the opportunity cost of providing a unit of capital. Thus, the demand for each type of bank loans determines the equilibrium quantity. Let $N_t$ denote the total bank capital in period $t$. The total wealth of households have in period $t$ is $q_t K_t - N_t$ and, by the Law of Large Numbers, the total wealth of experts is $α(q_t K_t - N_t)$. Hence, the total bank loans issued in equilibrium denoted by $L_t$ satisfies

$$L_t = α(q_t K_t - N_t) \int_{λ_{\text{min}}}^{λ_{\text{max}}} l^λ_t dG(λ),$$

where $l^λ_t$ denotes the debt-to-equity ratio for type $λ$ firms that receive bank loans.

The demand for final goods consists of consumption, intermediation costs, and investments. The aggregate consumption of households is $(q_t K_t - N_t)/ρ$, and bankers’ consumption is zero when the marginal value of bank capital $θ_t$ is larger than 1. The total intermediation cost is $τL_t$. Therefore, when $θ_t > 1$ the market clearing condition with respect to final goods is

$$ρ(q_t K_t - N_t) + τL_t = (1 - α) \frac{q_t K_t - N_t}{q_t} (a_ h - ι_t) x_t$$

$$+ α \frac{q_t K_t - N_t}{q_t} \left( \int_{λ_{\text{min}}}^{λ_{\text{max}}} (a - ι_t)(1 + b^λ_t) dG(λ) + \int_{λ_{\text{min}}}^{λ_{\text{max}}} (a - ι_t)(1 + l^λ_t) dG(λ) \right),$$

where $b^λ_t$ is the debt-to-equity ratio for type $λ$ firms that issue corporate debt. Finally, the market for physical capital clears if

$$(1 - α) \frac{q_t K_t - N_t}{q_t} x_t + α \frac{q_t K_t - N_t}{q_t} \left( \int_{λ_{\text{min}}}^{λ_{\text{max}}} 1 + b^λ_t dG(λ) + \int_{λ_{\text{min}}}^{λ_{\text{max}}} 1 + l^λ_t dG(λ) \right) = K_t.$$
3.4 Wealth Distribution

The share of total banking capital in the economy’s total wealth matters for the equilibrium. The endogenous state variable that will characterize the dynamics of the economy is bankers’ wealth share

\[ \eta_t = \frac{N_t}{q_t K_t}. \]  

(20)

If bankers’ wealth share declines, then the supply of bank loans shrinks and the interest rate on bank loans rises, which, in turn, lowers the proportion of experts raising external finance as well as the aggregate productivity of the economy.

Dynamics of the state variable in equilibrium also depend on the law of motion of the aggregate physical capital, which is

\[ \frac{dK_t}{K_t} = \mu^K_t dt + \sigma_d Z_t, \text{ where} \]

\[ \mu^K_t \equiv \Phi(\iota_t) - \delta - (1 - \alpha)(1 - \eta_t)x_t(\delta - \delta^h) - \alpha(1 - \eta_t) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda(1_{b_0 > \theta} b^h_0 \lambda^d + l_0^\lambda \kappa) d\lambda. \]

Given the law of motion of \( N_t, q_t, \) and \( K_t, \) we can apply Ito’s Lemma to derive the state variable’s law of motion in equilibrium, which is summarized in the following lemma.

**Lemma 1** In equilibrium, when \( \theta_t > 0 \) the state variable \( \eta_t \) evolves according to

\[ \frac{d\eta_t}{\eta} = \mu^K_t dt + \sigma^q_t dZ_t, \]  

(22)

where

\[ \eta \mu^K_t = -\alpha(1 - \eta_t)(\sigma^K_t (\sigma + \sigma^q_t) + (\sigma + \sigma^q_t)^2) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda l_t^\lambda dG(\lambda) + \eta_t r_t - \mu^q_t - \mu^K_t - \sigma^q_t \]

\[ \eta \sigma^q_t = (\sigma + \sigma^K_t)(\alpha(1 - \eta_t) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda l_t^\lambda dG(\lambda) - \eta_t) \]

The proof of Lemma 1 is in appendix.

3.5 Equilibrium without Banking

In the extreme case of our economy where the banking sector has no capital, the price of physical capital is constant since the wealth share of productive experts is the same across the time and the productivity shock that households receive is independent. Hence, \( \mu^K_t = 0 \) and \( \sigma^q_t = 0 \) for \( t \geq 0. \) To characterize the equilibrium, we look for a pair of two constants \((q, r)\)
such that both the market of physical capital goods (condition (19)) and the market of final goods (condition (18)) clear given the consumption and portfolio choices of all households determined by equation (9), (12), and (14). The price of physical capital in this equilibrium is denoted by $q$.

When the capital of the banking sector is sufficient, experts with high liquidity risk $\lambda$, who choose not to issue corporate debt due to the high liquidation cost $\lambda \kappa d$, expand their production through external credit from bank loans. Accordingly, the aggregate productivity improves compared to the zero bank capital case, and the market clearing condition of final goods (18) implies that the price of physical capital will be definitely higher than it is in the extreme case, $q$.

### 3.6 Markov Equilibrium

Like other continuous-time macro-finance models (Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2012), our framework also has the property of scale-invariance with respect to total physical capital $K_t$. Thus, we will focus on the equilibrium that is Markov in the state variable $\eta_t$. In the Markov equilibrium, dynamics of endogenous variables such as $q_t$ and $\theta_t$ can be characterized by the law of motion of $\eta_t$ and functions $q(\eta)$ and $\theta(\eta)$.

To solve for full dynamics of the economy, we derive a system of differential equations with respect to $q(\eta)$ and $\theta(\eta)$ based on equilibrium conditions. The following proposition defines the system of differential equations.

**Proposition 2** In the Markov equilibrium, $q(\eta)$ and $\theta(\eta)$ are defined over $(0, \eta^*)$, where $\eta^*$ is the reflecting boundary for the stochastic process $\{\eta_t, t \geq 0\}$. Experts’ consumption is zero when $\eta_t < \eta^*$ and positive when $\eta_t = \eta^*$. Boundary conditions for differential equations are

$$
q(0) = q, \quad \lim_{\eta \to 0} \theta(\eta) = \infty, \quad \theta(\eta^*) = 1, \quad q'(\eta^*) = 0, \quad \text{and,} \quad \theta'(\eta^*) = 0.
$$

We use the backward Euler method to derive $(q''(\eta), \theta''(\eta))$. Given that $d\eta$ is an arbitrarily small number, our procedure takes $\{q(\eta - d\eta), q'(\eta - d\eta), \theta(\eta - d\eta), \theta'(\eta - d\eta)\}$ as inputs and calculate $\{q(\eta), q'(\eta), q''(\eta), \theta(\eta), \theta'(\eta), \theta''(\eta)\}$ through three layers of calculations that rely on the clearing condition of the physical capital market, the identity for bankers’ exposure to the aggregate risk, and the clearing condition of the final good market:

1. the bottom layer of calculation takes $\{q(\eta), \sigma^q, \sigma^\theta\}$ as given, and produces households’ portfolio choices $\{x^h, b^\lambda, l^\lambda\}$ based on equation (11), (12), (13), (14), the banking sector’s exposure to the aggregate risk $\int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda d\lambda$ according to the identity (23) derived in the proof of Lemma 1, the drift of the state variable $\eta \mu^n$ based on equation (22), and $r - \mu^\theta$, $\mu^\theta$, $\mu^n$. 

15
which clears the market of physical capital;
2. the intermediate layer of calculation takes \( \{q(\eta), q'(\eta), \theta(\eta), \theta'(\eta)\} \) as given, uses formula
\[
\eta \sigma^q = \frac{\left( \eta \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda d\lambda - \eta \right) \sigma}{1 - \left( \eta \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda d\lambda - \eta \right) \frac{\eta}{q(\eta)}},
\]
\[
\sigma^q = \frac{q'(\eta)}{q(\eta)} \eta \sigma^q, \text{ and, } \sigma^\theta = \frac{\theta'(\eta)}{\theta(\eta)} \eta \sigma^q
\]
to calculate \( \{\eta \sigma^q, \sigma^q, \sigma^\theta, \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda d\lambda\} \), of which \( \{\sigma^q, \sigma^\theta\} \) is passed to the bottom layer and \( \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda d\lambda \) equals the same term that the bottom layer generates.
3. the upper layer of calculation generates \( \{q(\eta), q'(\eta), q''(\eta), \theta(\eta), \theta'(\eta), \theta''(\eta)\} \) based on
\[
q'(\eta) = q'(\eta - \phi \eta) + \phi \eta q''(\eta), \quad q(\eta) = q(\eta - \phi), \quad q(\eta) = q(\eta - \phi \eta) + \phi \eta q''(\eta),
\]
\[
\theta'(\eta) = \theta'(\eta - \phi \eta) + \phi \eta \theta''(\eta), \quad \theta(\eta) = \theta(\eta - \phi \eta) + \phi \eta \theta''(\eta)
\]
\[
q_{\mu} = q'(\eta) \eta \mu + 0.5 q''(\eta) (\eta \sigma^q)^2, \quad \text{and, } \theta \mu \theta = \theta'(\eta) \eta \mu + 0.5 \theta''(\eta) (\eta \sigma^q)^2
\]
of which \( \{q(\eta), q'(\eta), \theta(\eta), \theta'(\eta)\} \) is passed to the intermediate and bottom layers and \( \{q''(\eta), \theta''(\eta)\} \) is such that the market of final goods clears given portfolio choices of both households and bankers produced in the bottom layer.

The algorithm used to solve for the equilibrium relies on Proposition 2. We use the backward Euler method to calculate \( (q(\eta), \theta(\eta), q'(\eta), \theta'(\eta), q''(\eta), \theta''(\eta)) \) simultaneously. The following section will illustrate key properties of the model with numerical examples.

4 Results

In this section, we discuss main results of the model with numerical examples. The choice of parameter values is \( \rho = 3\%, \ a = 0.275, \ a^b = 0, \ d = -0.05, \ d^h = 0, \ \phi = 5, \ \tau = 5\%, \ \alpha = 5\%, \ \lambda_{\min} = 0, \ \lambda_{\max} = 0.6, \ \kappa^d = 0.6, \ \kappa = 0.1, \ \sigma = 0.1, \ \text{and } G(\lambda) = \frac{\lambda - \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}.

4.1 Price and the Misallocation of Physical Capital

The price of physical capital converges to its lower bound \( q = 1.517 \) when the share of bankers’ net worth is arbitrarily close to zero. As the banking sector becomes more capitalized, the price of physical capital appreciates and the misallocation of physical capital becomes less severe (Plot a and c in Figure 1). The price of physical capital is lower in
the economy without banking than it is in the economy with banking because 1) more productive experts cannot issue outside equity to normal households, and 2) bond-financing involves more costly liquidation than bank loans do.

Plot b in Figure 1 indicates that the marginal value of bankers’ net worth rises when the economy is losing its bank capital. The underlying reason of this result is related to the limited supply of bank capital, which only comes from bankers’ net worth since banks cannot issue outside equity. On the one hand, bankers can only accumulate bank capital with retaining their earnings; on the other hand, the use of leverage exposes bank capital to a large proportion of aggregate risks. When adverse aggregate shocks deplete bank capital, the decreased supply of bank loans leads to the rise of interest rates that banks demand, which in turn raises the marginal value of bank capital.

![Figure 1](image)

**Figure 1:** $q$, $\theta$, and $\psi$ as functions of the state variable $\eta$ in equilibrium. For parameter values, see the beginning of Section 4.

Let $\psi_t$ denote the fraction of physical capital held by experts, which satisfies

$$\psi_t = 1 - (1 - \eta_t)(1 - \alpha)x_t,$$

according to equation (19). As bankers’ wealth share rises, the supply of bank loans increases and more experts find it optimal to raise external funds and hold physical capital. By rearranging the clearing condition of the final good market for the case that $\theta_t > 1$, we have

$$q_t(1 - \eta_t)(\rho + \tau_\alpha \int_{\lambda_{\min}}^{\lambda_{\max}} l_t^1 \lambda dG(\lambda)) = \psi_t(a - \iota_t) + (1 - \psi_t)(a_h - \iota_t).$$
The above equation implies that as experts possess more and more more physical capital in the economy, the aggregate productivity rises and the price of physical capital also increases.

4.2 Stationary Distribution

In the long run, the economy stays in states close to where the banking sector is well capitalized such that banks pay out dividends, that is, \( \theta_t = 1 \). When banks are undercapitalized, the limited supply of bank loans raises interest rates that banks demand, and high returns from financial intermediation help banks quickly accumulate net worth. This is why the economy rarely moves to states where bankers’ wealth share is extremely low, as the density function of the state variable’s stationary distribution shows in Figure 2.

![Density of stationary distribution](image)

**Figure 2:** Density of stationary distribution in equilibrium. For parameter values, see the beginning of Section 4.

4.3 Endogenous Risk and Amplification Mechanism

The exogenous Brownian shocks hit bank capital in the economy due to the fact that banks will seize physical capital funded by their lending \( x^\lambda_t \) if liquidity shocks hit their \( \lambda \) type borrowers in period \( t \). This effect is amplified by the financial intermediary sector through the following vicious spiral. The decline in bank capital raises the cost of obtaining bank loans and deter firms from raising external funds. And, this lowers the aggregate productivity and pushes down the price of physical capital, which in turn impairs bank capital further.

Suppose an adverse exogenous Brownian shock causes that the stock of physical capital declines by 1 percent, the immediate total loss to bankers is \( \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda_t d\lambda \) percent of their total net worth \( N_t \). Accordingly, the immediate effect of the exogenous Brownian shock on the state variable is that \( \eta_t \) declines by \( \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x^\lambda_t d\lambda - 1 \) percent. As plot c in Figure 1 shows,
the decrease in the share of bank capital in the economy causes that the price of physical capital declines by

\[
\frac{q'(\eta_t)}{q(\eta_t)} \left( \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda - 1 \right) \eta_t.
\]

The decline in the price of physical capital gives rise to further losses to bank capital because once banks seize physical capital from firms, they need to sell these assets to firms, and banks will take losses during this process. As in Brunnermeier and Sannikov (2014), the overall impact of the initial adverse shock on the state variable is

\[
\frac{\eta_t \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda - \eta_t}{1 - \left( \eta_t \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda - \eta_t \right) \frac{q'(\eta_t)}{q(\eta_t)}},
\]

and the overall impact on the price of physical capital is

\[
q'(\eta_t) \frac{\eta_t \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda - \eta_t}{1 - \left( \eta_t \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda - \eta_t \right) \frac{q'(\eta_t)}{q(\eta_t)}},
\]

which equals the magnitude of the endogenous risk \( q_t \sigma^q \).

### 4.4 Dynamics of Price Variables

If a series of negative exogenous shocks hit the economy, the financial health of the banking sector deteriorates and the price of physical capital declines. As a result, the expected return from holding physical capital for experts goes up when the economy is in economic downturns (Plot a in Figure 3).

The risk-free rate is very low when the economy is in recessions, where the banking sector is deeply undercapitalized (Plot b in Figure 3). The underlying reason is that the excessive supply of credit depress the risk-free rate when the banking sector almost stops functioning. We will later explain why the risk-free rate displays a hump-shape in dynamics later when we discuss the substitution between bond-financing and bank-financing.

The magnitude of endogenous risks fluctuates in this dynamic economy. \( \sigma^q \) (the percentage change in the price of physical capital caused by exogenous shocks) is low when the well capitalized banking sector can easily cope with adverse shocks. However, endogenous risks are also small when the banking sector is terribly undercapitalized. The reason is that when the supply of bank loans is negligible compared to outstanding corporate debt any change in the amount of bank loans has a minimum impact on the relative holdings of physical capital.
Figure 3: expected excess return $R - r$, risk-free rate $r$, aggregate risk $\sigma^q$, and the threshold for bank financing $\bar{\lambda}$ as functions of the state variable $\eta$ in equilibrium. For parameter values, see the beginning of Section 4.

and the movement of its price. We also delay the explanation of the hump-shape property of endogenous risks to the discussion about the substitution between bond-financing and bank-financing.

4.5 Endogenous Fluctuation of Intermediation Costs

Costs of both bond-financing and bank-financing consist of two components: the cost of liquidation and the interest rate charged by creditors. Bank-financing dominates bond-financing in terms of the cost of liquidation, $\lambda \kappa < \lambda \kappa^d$. This effect is especially large for firms with high liquidity risks. With respect to the interest payment, firms only pay the
risk-free rate for issuing corporate debt regardless of their liquidity risks. In contrast, raising credit from banks involves compensating banks for their exposures to both exogenous risk and endogenous risk, \(-\sigma_t^\theta(\sigma + \sigma_t^\theta)\) as well as the unit intermediation cost \(\tau\). Recall

\[ r_t^\lambda = r_t + \tau - \sigma_t^\theta \lambda (\sigma + \sigma_t^\theta), \text{ for } x_t^\lambda > 0. \]

One particular feature of bank-financing in our model is that its cost fluctuates endogenously in the dynamics of the economy. Dynamics of the cost of bank-financing depend on three components: banks’ risk appetite, i.e., the sensitivity of the marginal value of bank capital to exogenous risks, \(\sigma_t^\theta\), the liquidity risk of a particular firm \(\lambda\), and the magnitude of endogenous risk \(\sigma_t^q\). When the banking sector is well capitalized, it is relatively resilient to adverse exogenous shocks. Hence, both \(\sigma_t^\theta\) and \(\sigma_t^q\) are small in economic booms, and thus firms especially those with high liquidity risks find it more profitable to raise credit from banks in economic upturns. In downturns, however, when the banking sector is not financially healthy, banks become less tolerant of taking risks and endogenous risks also go up. Overall, the rise in the cost of bank-financing in downturns squeezes firms with high liquidity risks to more costly bond-financing or self-financing, which of course hurts the aggregate productivity.

4.6 Firms’ Financing Choice

The financing choice of a firm relies on its liquidity risk \(\lambda\). Firms with relatively low liquidity risks are inclined to choose bond-financing. To draw a more concrete conclusion, we recall \(G_t^b\) and \(G_t^l\) defined in Section 3.1.1. It is straightforward to see that the two problems belong to a family of optimization problem parametrized by the unit borrowing cost denoted by \(\mathcal{R}\)

\[ G_t \equiv \max_{x \geq 0} \left\{ R_t + x(R_t - \mathcal{R}_t) - 0.5(x(1 - \lambda_t))^2(\sigma + \sigma_t^q)^2 \right\}. \]

The Envelope Theorem implies that

\[ \frac{\partial G_t}{\partial \mathcal{R}_t} = -x^* \leq 0, \]

where \(x^*\) is the maximizer of the above optimization problem. Therefore, if the borrowing cost of bond-financing \(\lambda \kappa + r_t\) is lower than the borrowing of bank-financing \(\tau + \lambda \kappa + r_t^\lambda\) for a firm with liquidity risk \(\lambda\), then it will choose bond-financing. Note that the liquidity problem that a firm has may result in a fundamental problem with the firm’s asset, as we illustrate in footnote 1. Therefore, the liquidity risk can be interpreted as the signal of a
firm’s credit problem. Hence, our result is consistent with the empirical finding in Rauh and Sufi (2010) that firms with high credit ratings almost only rely on corporate debt and equity. Proposition 3 characterizes under what condition a firm will choose bond-financing.

**Proposition 3** Suppose $\kappa^d > \kappa - \sigma^q_t(\sigma + \sigma^q_t)$ in equilibrium, a firm chooses bond-financing in period $t$ if

$$R_t - \lambda_t \kappa^d - r_t - (1 - \lambda_t)(\sigma + \sigma^q_t)^2 > 0,$$

and its liquidity risk $\lambda_t < \bar{\lambda}_t$, where

$$\bar{\lambda}_t \equiv \frac{\tau}{\kappa^d - (\kappa - \sigma^q_t(\sigma + \sigma^q_t))}.$$

Plot d in Figure 3 shows dynamics of the threshold $\bar{\lambda}_t$. In economic booms when bank loans are relatively cheap, only firms with rather low liquidity risks still choose bond-financing. In contrast, when the banking sector is undercapitalized, expensive bank loans force firms to choose alternative financing channels.

Firms refrain from raising external funds and only consider internal financing when they find it too costly to issue corporate bond or borrow from banks, that is,

$$R_t - \lambda_t \kappa^d - r_t - (1 - \lambda_t)(\sigma + \sigma^q_t)^2 < 0,$$

and

$$R_t - \lambda_t \kappa - r^\lambda_t - (1 - \lambda_t)(\sigma + \sigma^q_t)^2 < 0.$$

### 4.7 Intensive Margin and Extensive Margin

The amount of outstanding corporate debt relies on the proportion of firms issuing this financial instrument and the leverage of these firms. Upper panels in Figure 4 illustrate the changes at the intensive margin and extensive margin along the business cycle. At the intensive margin, the average debt-to-equity ratio for firms issuing corporate debt displays a U-shape. In economic booms, these firms take high leverage primarily because of low endogenous risks (recall Plot c in Figure 3). In recessions, the leverage of these firms is also high because of high returns from holding physical capital and low risk-free rates (recall Plot a and b in Figure 3). At the extensive margin, the proportion of firms that issue corporate debt is typically very stable. Only when the banking sector is dramatically undercapitalized would the proportion of bond-financing firms shoot up (Plot b in Figure 4).
Figure 4: $\bar{b} \equiv \frac{1}{G(\bar{\lambda}_b) - G(\lambda_b)} \left( \int_{G(\bar{\lambda}_b)}^{G(\lambda_b)} b^\lambda d\lambda \right)$, $G(\bar{\lambda}_b) - G(\lambda_b)$, $\bar{l} \equiv \frac{1}{G(\bar{\lambda}_l) - G(\lambda_l)} \left( \int_{G(\bar{\lambda}_l)}^{G(\lambda_l)} l^\lambda d\lambda \right)$, and $G(\bar{\lambda}_l) - G(\lambda_l)$ as functions of the state variable $\eta$ in equilibrium, where firms with liquidity risk $\lambda \in [\lambda_b, \bar{\lambda}_b]$ choose bond-financing and firms with liquidity risk $\lambda \in [\lambda_l, \bar{\lambda}_l]$ choose bank-financing. For parameter values, see the beginning of Section 4.

The average loan-to-equity ratio among firms using bank-financing is pro-cyclical. The underlying driving force is related to the endogenous component of the intermediation cost $-\sigma^\theta_t (\sigma + \sigma_q^\eta)$. In recessions when the banking sector is short of capital, the rising borrowing cost $r^\lambda_t$ substantially lower the leverage of firms that still choose bank-financing because of their high liquidity risks. At the extensive margin, it is not surprising to see that the fraction of firms borrowing bank loans significantly declines when bankers’ wealth share decreases drastically in economic downturns.
4.8 Heterogeneity of Bond-Financing and Bank-Financing

Bond-financing is acyclical in our model. As the economy evolves into economic booms, the share of outstanding corporate debt in total wealth slightly goes up. This is primarily the consequence of bond financing firms’ high debt-to-equity ratio due to low endogenous risks. Our paper highlights that the credit market of direct finance can also benefit from the development of the financial intermediary sector. In economic downturns, the share of corporate debt is also high because 1) more firms switch to bond-financing due to the rising cost of bank-financing, and 2) firms take high leverage as a result of both high returns from holding physical capital and low risk-free rates.

![Figure 5](image-url)

**Figure 5**: share of bond financing \((1 - \eta)\alpha \int b_{(b>0)}^\lambda dG(\lambda)\), share of bank financing \((1 - \eta)\alpha \int l^\lambda dG(\lambda)\), share of total credit \((1 - \eta)\alpha \int (b_{(b>0)}^\lambda + l^\lambda) dG(\lambda)\), and share of internal financing \((1 - \eta)\alpha \int (b_{(b<0)}^\lambda + 1) dG(\lambda)\), and banks’ exposure to the aggregate risk \(\eta \int \lambda x^\lambda d\lambda\) as functions of the state variable \(\eta\) in equilibrium. For parameter values, see the beginning of Section 4.
In contrast, Panel b in Figure 5 shows that bank-financing is clearly pro-cyclical. This is true since bank-financing is pro-cyclical at both intensive margin and extensive margin (Panel c and d in Figure 4). In addition, the sharp decline of bank-financing in economic downturns is accompanied by the rise in share of physical capital that is solely financed by firms’ internal capital (Panel c in Figure 5).

Overall, our model accounts for two facts of bond-financing and bank-financing in business cycles. The first fact is that bank-financing is more volatile and cyclical than bond-financing in the long-run as Becker and Ivashina (2014) document. The second fact, which Adrian et al. (2012) and many other papers have highlighted, is that the drastic decline in intermediated finance during big recessions such as 2007-09 financial crisis is partially made up by the increase in direct finance.

The reason why our model can capture the two facts has to do with two features of our framework: a feature on the technical side and a feature on the economics side. The technical feature is that our continuous-time frame allows for the full characterization of the dynamics of the economy. Thus, we do not only know the property of the equilibrium around the steady state but also we can precisely observe the equilibrium outcome in extreme states. Sometimes, properties of the equilibrium could be quite different in different states of the economy as we have noticed in our framework.

The other feature is that our framework highlights the dynamics of endogenous risks and these dynamics have substantial effects on the dynamics of bond-financing. In particular, as the banking sector becomes more and more financially healthy, endogenous risks becomes lower and lower, which in turn actually help firms issuing more corporate debt. This result implies the outstanding corporate debt in the economy is not monotonic in the state of the economy.

### 4.9 Price Effects of Replacing Bank Loans with Corporate Debt

The substitution of bond credit for bank credit in economic downturns has significant price effects in equilibrium. When bank loans are very expensive, firms with relatively high liquidity risks have to replace bank credit with bond-financing. Noticing that bond-financing involves more costly liquidation than bank-financing does, the rising borrowing cost for firms exerts downward pressure on the price of physical capital. This explains why the magnitude of endogenous risks goes up when a large proportion of firms replacing bank credit with bond credit (Panel c in Figure 3). In addition, the rising demand for direct finance also pushes up the risk-free rate as Panel b in Figure 3 shows. And, this result is consistent with the observation that borrowing costs of both bond-financing and bank-financing rise as the
economy gradually evolves into recessions with the banking sector being undercapitalized.

### 4.10 Financial Intermediaries

Bankers who are financial intermediaries in the economy channel funds provided by normal households to more productive experts. However, financial intermediaries cannot issue outside equity to normal households due to asymmetric information problem modelled in papers such as He and Krishnamurthy (2012) and Brunnermeier and Sannikov (2014). As a result, bankers can only issue risk-free debt to normal households. The interest rate spread $r_t^\lambda - r_t$ that financial intermediaries earn from loans made to type $\lambda$ experts is composed of the intermediation cost $\tau$ and the risk-premium $-\lambda \sigma_t^\theta (\sigma + \sigma_t^q)$. Since the only source of bank capital is retained earning, banks will refrain pay out dividends until the marginal value of their capital equals one.

\[
\text{banks’ exposure}
\]

![Graph (a)](image)

\[
\text{bank leverage}
\]

![Graph (b)](image)

**Figure 6:** banks’ exposure to the aggregate risk $\int \lambda x^\lambda d\lambda$ and banks’ leverage $\int x^\lambda d\lambda$ as functions of the state variable $\eta$ in equilibrium. For parameter values, see the beginning of Section 4.
Figure 6 shows that both banks’ exposure to aggregate risks and banks’ asset-to-equity ratio are counter-cyclical. The second fact is standard in the literature. Banks’ exposure to aggregate risks can also be interpreted as banks’ risk-weighted asset-to-equity ratio, where the risk weight is naturally the liquidity risk of firms that borrow bank loans.

5 Conclusion

In this paper, we present a dynamic general framework, in which firms choose either bond-financing or bank-financing and banks channel credit from savers to borrowers. The intermediation cost of bank-financing fluctuates endogenously because the risk-premium that banks ask for depends on the financial health of the banking sector. Our model accounts for two seemingly-conflicting facts about bond-financing and bank-financing: although bank financing is more cyclical than bond-financing in the long run, firms especially those who have relatively high credit ratings replace bank loans with corporate debt during financial crises such as the 2007-09 Great Recession.

One natural extension of our current model is to have three groups of agents explicitly: experts, households, and bankers, and to investigate the interaction between dynamics of experts’ net worth and bankers’ net worth as well as how this interaction affects the equilibrium.

References


Appendix

A  A Household’s Dynamic Optimization Equation

To formalize a household’s dynamic optimization problem, we conjecture that the continuation value of a household is \( \frac{\ln(w_t)}{\rho} + h_t \), where \( \{h_t, t \geq 0\} \) follows

\[
\frac{dh_t}{h_t} = \mu_t^h dt + \sigma_t^h dZ_t.
\]

The continuation value must satisfy the Hamilton-Jacobi-Bellman (HJB) equation

\[
\rho \left( \frac{\ln(w_t)}{\rho} + h_t \right) = h_t \mu_t^h + \max_{c_t, x_t, b_t, l_t} \left\{ \ln(c_t) - \frac{c_t}{\rho w_t} + \frac{1-\alpha}{\rho} \left( x_t R_t^h + (1-x_t) r_t - 0.5 x_t^2 \sigma_t^h + (1-x_t) \sigma_t^q \right) \right\},
\]

where \( R_t^h, G_t^l, \) and \( G_t^i \) are defined in Section 3.1.1. First order conditions are listed in Section 3.1.

We can characterize the process \( \{h_t, t \geq 0\} \) in the Markov equilibrium by solving for \( h(\eta) \).
over \([0, \eta^*]\) such that \(h_t = h(\eta_t)\). The HJB equation yields the value of \(\mu_t^h\) given the state \(\eta_t\) and the value of \(h_t\). Combining this result with what Ito’s Lemma gives rise to

\[ h(\eta_t)\mu_t^h = h'(\eta_t)\eta_t\mu_t^h + 0.5h''(\eta_t)(\eta_t\sigma_t^\eta)^2, \]

we find the differential equation that \(h(\eta)\) must obey. The HJB equation at \(\eta_t = 0\) yields \(h(0)\) because the differential equation implies that \(\mu_t^h = 0\) at \(\eta_t = 0\). The second boundary condition is that \(h'(\eta^*) = 0\) since \(\eta^*\) is the reflecting boundary.

**B Proofs**

**Proof of Lemma 1.** The market clearing condition for each type of bank loans implies that

\[ x_t^\lambda N_t d\lambda = \alpha(q_t K_t - N_t)l_t^\lambda dG(\lambda). \]

Therefore,

\[ N_t \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda = \alpha(q_t K_t - N_t) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda l_t^\lambda dG(\lambda). \]

(23)

Base on the law of motion for individual banker’s net worth, we have the law of motion of \(N_t\).

\[
\frac{dN_t}{N_t} = \int_{\lambda_{\min}}^{\lambda_{\max}} x_t^\lambda (r_t^\lambda dt + \lambda(\sigma + \sigma_t^q) dZ_t - \tau) d\lambda + \left(1 - \int_{\lambda_{\min}}^{\lambda_{\max}} x_t^\lambda d\lambda \right) r_t dt,
\]

Given that the borrowing cost satisfies \(r_t^\lambda = r_t + \tau - \sigma_t^h(\sigma + \sigma_t^q)\), the above equation can be simplified as

\[
\frac{dN_t}{N_t} = \left(- \sigma_t^h(\sigma + \sigma_t^q) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda + r_t \right) dt + \left((\sigma + \sigma_t^q) \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda x_t^\lambda d\lambda \right) dZ_t
\]

To apply Ito’s Lemma, we have

\[
d(q_t K_t) = q_t K_t (\mu_t^q + \mu_t^K + \sigma \sigma_t^q) dt + q_t K_t (\sigma + \sigma_t^q) dZ_t.
\]
To apply Ito’s Lemma again, we can derive the law of motion of $\eta_t$

$$
d\eta_t = \frac{N_t}{q_t K_t} \left( -\sigma^2 (\sigma + \sigma^2) \int_{\lambda_{min}}^{\lambda_{max}} \lambda x_t^\lambda d\lambda + r_t \right) dt - \frac{N_t}{q_t K_t} (\mu^q_t + \mu^K_t + \sigma q_t^2) dt \\
- \frac{N_t}{q_t K_t} \left( (\sigma + \sigma^2)^2 \int_{\lambda_{min}}^{\lambda_{max}} \lambda x_t^\lambda d\lambda \right) dt + \frac{N_t}{q_t K_t} (\sigma + \sigma^2) \left( \int_{\lambda_{min}}^{\lambda_{max}} \lambda x_t^\lambda d\lambda - 1 \right) dZ_t \\
= \left( -\alpha (1 - \eta_t) (\sigma^2 (\sigma + \sigma^2) + (\sigma + \sigma^2)^2) \int_{\lambda_{min}}^{\lambda_{max}} \lambda t^\lambda dG(\lambda) + \eta_t (r_t - \mu^q_t - \mu^K_t - \sigma q_t^2) \right) dt \\
+ (\sigma + \sigma^2) \left( \alpha (1 - \eta_t) \int_{\lambda_{min}}^{\lambda_{max}} \lambda t^\lambda dG(\lambda) - \eta_t \right) dZ_t
$$

The last equality comes from equation (23). □