A Probabilistic Solution to High-Dimensional Continuous-Time Macro and Finance Models

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May 2023

Large Model in Economics

- Economists have been interested in Large Models since 1940's
  Lawrence Klein (Nobel Laureate 80')
- Lucas critique, Rational Expectation, Dynamic Programming
  Robert Lucas (Nobel Laureate 95')
- Curse of dimensionality
  Current models: either small or linearized local solution.
- This project intends to develop a general numerical approach that
  breaks the curse of dimensionality and
  revives the building of large models in economics.

Deep Learning-Based Probabilistic Approach

Implementation almost the same as Monte Carlo simulations;

Job specialization: *Economists*, *computer scientists*, and *mathematicians*

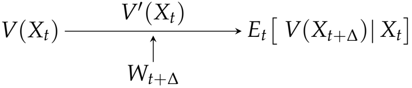
Applicable to

- HANK, Asset Pricing in HANK
- Heterogeneous Firms, Firm Dynamics, Knowledge Diffusion
- Asset Pricing in incomplete markets with heterogeneous investors
- Money search, Asset Pricing in OTC markets, labor search
- Trade, Dynamic firm entry, Dynamic spatial GE

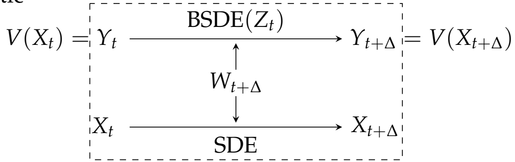
  (in progress)
- Limited commitment, hidden action and information
- Ramsey problem

What is Probabilistic Approach?

Analytic



Probabilistic



BSDE: the projection of $Y_{t+\Delta} - Y_t$ on Δ and $W_{t+\Delta} - W_t$,
where Z_t is the coefficient of $W_{t+\Delta} - W_t$.

Deep learning is used to approximate $V(\cdot)$ with $\tilde{V}(\cdot;\Theta)$
by minimizing $\sum_i \left(Y^i_{t+\Delta} - \tilde{V}(X^i_{t+\Delta};\Theta)\right)^2$ given $Y^i_t = \tilde{V}(X^i_t;\Theta)$

How does Deep Learning Help?

A layman's perspective: parametric approximation

- large sample is used to "learn" the property on a large state space
  value function, policy function, etc.
- mini-batch:
  cut the sample into a *random* sequence of small blocks
- automatic differentiation
- stochastic gradient descent and its variants:
  update parameter based on the average info of small blocks.

Backward Stochastic Differential Equation

Suppose $W_{t+\Delta} - W_t \sim N(0, \Delta)$,

A forward SDE is

$$X_{t+\Delta} = X_t + \mu(X_t)\Delta + \sigma(X_t)(W_{t+\Delta} - W_t)$$
$$X_0 = x$$

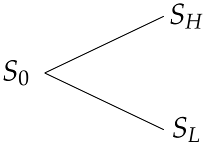
The solution of a forward SDE is a family of trajectories $\{X_t\}$

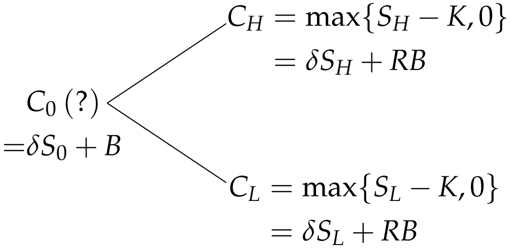
Given a r.v. ξ, whose value is realized at time T, a backward SDE

$$Y_{t+\Delta} = Y_t + \mu(Y_t)\Delta + Z_t(W_{t+\Delta} - W_t)$$
$$Y_T = \xi$$

The solution of a BSDE is a pair $\{Y_t, Z_t\}$.

BSDE: A Binomial Tree Illustration





The solution pair: $\{C_0, \delta\}$

Some Important Names

Andrey Kolmogorov: transition prob over state space (analytic)

Kiyosi Ito: SDE, dynamics of a stochastic path (probabilistic)

Peng Shige: BSDE and stochastic maximum principle

Zhou Xun Yu on
the connection between dynamic programming and
stochastic maximum principle

Ma Jin and Yong Jiongmin on
the solution of BSDE as an optimal control problem

E Weinan, Han Jiequn, Arnulf Jentzen on
solving BSDE with deep reinforcement learning

Plan

- Probabilistic approach: an oversimplified illustration
- Algorithm
- How to formulate Forward-Backward SDEs
  - recursive utility
  - dynamic optimization
  - homothetic preference with linear budget constraint
  - asset pricing
  - finite volume method
- Multiple-country macro-finance example
- Heterogeneous-agent macro example

Searching for the Fixed Point

Suppose X_t a Markov process,

$$V\left(X_t\right) = u\left(X_t\right)\Delta + E\left[V\left(X_{t+\Delta}\right) \middle| X_t\right] \tag{1}$$

Given fixed Δ, we need

- the info of $V(\cdot)$ over the entire state space;
- the transition probability $P\left(X_{t+\Delta} \middle| X_t = x\right)$.

to compute $V(\cdot)$ at x.

In Continuous-Time Setting

Suppose

$$X_{t+\Delta} = X_t + \mu\left(X_t\right)\Delta + \sigma\left(X_t\right)\left(W_{t+\Delta} - W_t\right)$$
$$W_{t+\Delta} - W_t \sim N(0, \Delta)$$

Rearranging equation (1) and apply Taylor expansion

$$\begin{aligned} 0 &= u\left(x\right) + E\left[\left.\frac{V\left(X_{t+\Delta}\right) - V\left(X_t\right)}{\Delta}\right| X_t = x\right] \\ &= u\left(x\right) + V'\left(x\right)\mu\left(x\right) + \frac{1}{2}V''\left(x\right)\sigma^2\left(x\right) \end{aligned}$$

When $\Delta \to 0$, we can make use of (stochastic) *calculus*.

Solving Differential Equation

$$
\begin{aligned}
0 = u(x) &+ \frac{V(x+k) - V(x-k)}{2k}\mu(x) \\
&+ \frac{V(x+k) + V(x-k) - 2V(x)}{2k^2}\sigma^2(x), k > 0
\end{aligned}
$$

To compute $V(\cdot)$ at x, we only need

two evaluation points of $V(\cdot)$: $x - k$ and $x + k$.

What if X has infinite dimension, e.g., wealth distribution?

there are infinitely many directions even in the neighborhood of x.

Probabilistic Formulation

Rewrite equation (1)

$$E\left[V\left(X_{t+\Delta}\right)\middle|\, X_t = x\right] = V\left(x\right) - u\left(x\right)\Delta.$$

Recall

$$X_{t+\Delta} = X_t + \mu\left(X_t\right)\Delta + \sigma\left(X_t\right)\left(W_{t+\Delta} - W_t\right)$$

When $\Delta \to 0$, there exists $z(x)$ such that

$$V\left(X_{t+\Delta}\right) = V\left(x\right) - u\left(x\right)\Delta + z(x)\left(W_{t+\Delta} - W_t\right),$$

In stochastic calculus, this is the Martingale Representation Theorem.

Probabilistic Numerical Method

To solve for $V(x)$ and $z(x)$, we simulate two paths: $W^1_{t+\Delta}$ and $W^2_{t+\Delta}$, obtain $X^1_{t+\Delta}$ and $X^2_{t+\Delta}$ based on X_t's law of motion.

two equations:

$$V\left(X^1_{t+\Delta}\right) = V(x) - u(x)\,\Delta + z(x)\left(W^1_{t+\Delta} - W_t\right)$$
$$V\left(X^2_{t+\Delta}\right) = V(x) - u(x)\,\Delta + z(x)\left(W^2_{t+\Delta} - W_t\right),$$

and two evaluation points of $V(\cdot)$ at $X^1_{t+\Delta}$ and $X^2_{t+\Delta}$.

The calculation remains largely the same for high-dimensional X_t.

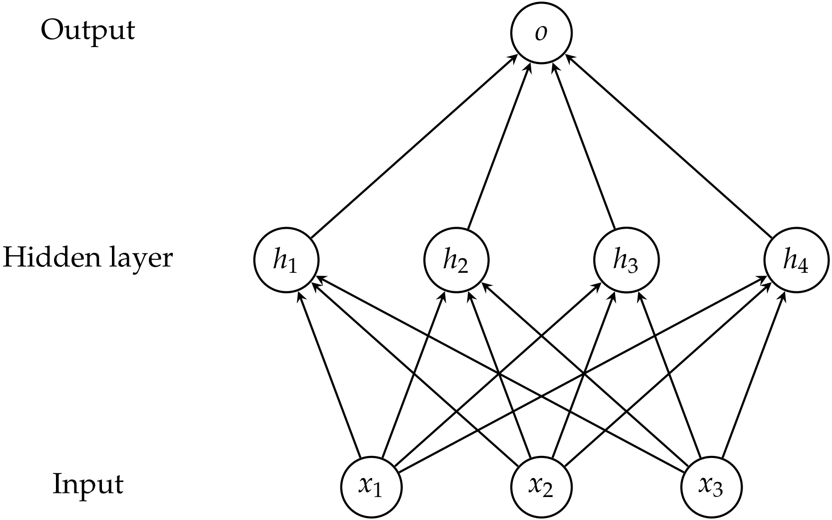
Deep Learning-Based Probabilistic Solution

To search for the parametric approximation of $V(\cdot)$ and $z(\cdot)$:

$\tilde{V}(\cdot;\Theta)$ and $\tilde{z}(\cdot;\Theta)$

$$
\begin{aligned}
\min_{\Theta} :\quad & \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\tilde{V}(\hat{x}^{i,j};\Theta) - \tilde{V}(x^{i};\Theta) + u(x^{i})\Delta - \tilde{z}(\cdot;\Theta)w^{i,j}\right)^{2} \\
\text{s.t.}\quad & \hat{x}^{i,j} = x^{i} + \mu(x^{i})\Delta + \sigma(x^{i})w^{i,j} \\
& w^{i,j} \text{ is sampled independently from } N(0,\Delta) \\
& x^{i} \text{ is sampled from a prior distribution}
\end{aligned}
$$

Neural Network: an Diagram



Neural Network as an Approximation

Input (state variables): x_1, x_2, x_3

Hidden-layer variables: h_1, h_2, h_3, h_4

$$h_i = \sigma\big(b_i^1 + w_{i,1}^1 x_1 + w_{i,2}^1 x_2 + w_{i,3}^1 x_3\big), \qquad i = 1, \cdots, 4,$$

$\sigma(u)$: activation function, e.g.,

$$max(u, 0), \quad \frac{1}{1 + \exp(-u)}, \quad \frac{1 - \exp(-2u)}{1 + \exp(-2u)}, \quad \text{etc}$$

Output variable: o

$$o = b^2 + w_1^2 h_1 + w_2^2 h_2 + w_3^2 h_3 + w_4^2 h_4.$$

$b_{i,k}^j, w_{i,k}^j$ are parameters to learn.

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Forward-Backward SDE System

Dynamics of an infinite horizon continuous-time model is characterized by a coupled Forward-Backward SDE system.

$$
\begin{aligned}
\mathrm{d}X_t &= b(X_t, Y_t, Z_t)\mathrm{d}t + \sigma(X_t, Y_t, Z_t)\mathrm{d}W_t \\
X_0 &= x_0 \\
-\mathrm{d}Y_t &= h(X_t, Y_t, Z_t)\mathrm{d}t - Z_t\mathrm{d}W_t
\end{aligned}
$$

X_t: stocks of physical capital, entrepreneurs' net worth, etc.

Y_t: the price of physical capital, entrepreneurs' life-time utility, etc.

Z_t: the volatility of capital price, the volatility of life-time utility, etc.

Markov Equilibrium

The Markov equilibrium is characterized by functions $y(\cdot)$ and $z(\cdot)$, where X_t state variable, $Y_t = y(X_t)$ and $Z_t = z(X_t)$

Given the time-homogeneity of the model, the solution of the following FBSDE $\{X_t, Y_t, Z_t\}_{0 \le t \le T}$

$$
\begin{aligned}
X_t &= x_0 + \int_0^t b(X_s, Y_s, Z_s)\mathrm{d}s + \int_0^t \sigma(X_s, Y_s, Z_s)\mathrm{d}W_s \\
Y_t &= y(X_T) + \int_t^T h(X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s.
\end{aligned}
$$

satisfies $Y_t = y(X_t)$ and $Z_t = z(X_t)$ for $0 \le t \le T$

Algorithm

$y(\cdot)$ and $z(\cdot)$ approximated by DNN $\tilde{y}(\cdot;\Theta)$ and $\tilde{z}(\cdot;\Theta)$

1. initialize sample $\{x_0^i, W_t^i, 0 \le t \le T\}_{i=1}^M$;
2. generate $\{X_t^i, Y_t^i, Z_t^i, 0 \le t \le T\}$ according to the FBSDE;
3. calculate the average loss $\left(Y_t^i - \tilde{y}(X_t^i;\Theta)\right)^2$ over M samples and time $0 \le t \le T$.

Minimize the loss function by choosing DNN parameter Θ with Machine Learning libraries like PyTorch or TensorFlow.

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Recursive Utility

Given $\{c_t\}$, Duffie and Epstein (1992) defines

$$V_t = E_t\left[\int_t^{\infty} f\left(c_s, V_s\right) \mathrm{d}s\right].$$

When $f(c, V) = c - \rho V$, it leads to the expected discounted utility.

In a Markov equilibrium with state variable X_t, the BSDE

$$-\mathrm{d}V_t = f\left(c\left(X_t\right), V_t\right) \mathrm{d}t - Z_t \mathrm{d}W_t$$

yields the value function $V(\cdot)$ by time-homogeneity $V_T = V(X_T)$.

Dynamic Optimization

an agent maximizes the objective function

$$V\left(X_0, X_0^0\right) = \max_{\{\alpha_t\}} : E_0\left[\int_0^\infty e^{-\rho s} f\left(X_s, X_s^0, \alpha_s\right) \mathrm{d}s\right]$$

subject to

$$\mathrm{d}X_t = \mu\left(X_t, X_t^0, \alpha_t\right)\mathrm{d}t + \sigma\left(X_t, X_t^0, \alpha_t\right)\mathrm{d}W_t + \sigma^0\left(X_t, X_t^0, \alpha_t\right)\mathrm{d}W_t^0$$
$$\mathrm{d}X_t^0 = b\left(X_t^0\right)\mathrm{d}t + \Sigma\left(X_t^0\right)\mathrm{d}W_t^0.$$

X_t^0: aggregate state; W_t^0: aggregate shock

X_t: individual state; W_t: idiosyncratic shock

Hamiltonian System

$$H\left(x, x^0, \alpha, y, z, z^0\right) = \\ \mu\left(x, x^0, \alpha\right) y + \sigma\left(x, x^0, \alpha\right) z + \sigma^0\left(x, x^0, \alpha\right) z^0 + f\left(x, x^0, \alpha\right)$$

The optimal control $\hat{\alpha}_t$ satisfies

$$\hat{\alpha}_t = \arg\max_{\alpha} : H\left(X_t, X_t^0, \alpha, Y_t, Z_t, Z_t^0\right),$$

where $\left(Y_t, Z_t, Z_t^0\right)$ follows the BSDE

$$-\mathrm{d}Y_t = \left(\frac{\partial}{\partial x} H\left(X_t, X_t^0, \hat{\alpha}_t, Y_t, Z_t, Z_t^0\right) - \rho Y_t\right) \mathrm{d}t - Z_t \mathrm{d}W_t - Z_t^0 \mathrm{d}W_t^0$$

$$Y_T = \frac{\partial}{\partial x} V\left(X_T, X_T^0\right)$$

Homothetic Preference I

$\cdots$ with a Linear Budget Constraint.

Investors have the Epstein-Zin preference

$$V_t = \mathbb{E}_t \left[\int_t^\infty f\left(c_u, V_u\right) \mathrm{d}u \right],$$

$$f\left(c, V\right) = \frac{1}{1-\psi} \left\{ \frac{\rho c^{1-\psi}}{\left[\left(1-\gamma\right) V\right]^{(\gamma-\psi)/(1-\gamma)}} - \rho\left(1-\gamma\right) V \right\}$$

with the budget constraint

$$\frac{\mathrm{d}N_t}{N_t} = \left(\mu\left(X_t, \alpha_t\right) - \hat{c}_t\right) \mathrm{d}t + \sigma\left(X_t, \alpha_t\right) \mathrm{d}W_t.$$

Homothetic Preference II

V_t has the functional form

$$V_t = \frac{\left(\xi_t N_t\right)^{1-\gamma}}{1-\gamma},$$

and ξ_t follows a BSDE

$$\mathrm{d}\xi_t = \xi_t \left(\mu_t^{\xi} \mathrm{d}t + \sigma_t^{\xi} \mathrm{d}W_t\right)$$

since V_t does so. Apply Ito's formula to V_t,

$$-\mu_t^{\xi} = \frac{1}{1-\psi}\left\{\rho\left(\frac{\hat{c}_t}{\xi_t}\right)^{1-\alpha} - \rho\right\} + \mu\left(X_t, \alpha_t\right) - \hat{c}_t + \sigma_t^{\xi}\sigma\left(X_t, \alpha_t\right)$$
$$- \frac{1}{2}\gamma\left(\sigma_t^{\xi}\right)^2 + (1-\gamma)\,\sigma_t^{\xi}\sigma\left(X_t, \alpha_t\right) - \frac{1}{2}\gamma\sigma^2\left(X_t, \alpha_t\right)$$

Asset Pricing I

Stochastic discount factor $m(X_t)$ follows

$$\frac{\mathrm{d}m_t}{m_t} = \mu^m(X_t)\,\mathrm{d}t + \sigma^m(X_t)\,\mathrm{d}W_t.$$

Consider an asset with dividend $D(X_t)$ with terminal payoff $g(X_T)$

$$m(X_t)\,p(t, X_t) = E\left[\int_t^T m(X_u)\,D(X_u)\,\mathrm{d}u + m(X_T)\,g(X_T)\,|X_t\right]$$

To construct a martingale,

$$\begin{aligned} & E\left[\int_0^T m(X_u)\,D(X_u)\,\mathrm{d}u + m(X_T)\,g(X_T)\,|X_t\right] \\ = & \int_0^t m(X_u)\,D(X_u)\,\mathrm{d}u + m(X_t)\,p(t, X_t) \end{aligned}$$

Asset Pricing II

Apply Martingale representation and Ito's formula, p_t follows a BSDE

$$-\frac{\mathrm{d}p_t}{p_t} = \left(\frac{D_t}{p_t} + \mu_t^m + \sigma_t^m \sigma_t^p\right)\mathrm{d}t - \sigma_t^p \mathrm{d}W_t.$$
$$p_T = g\left(X_T\right)$$

In the continuous-time macro-finance, the shortcut is to postulate the law of motion for the asset price $\{p_t\}_{t\geq 0}$

$$\frac{\mathrm{d}p_t}{p_t} = \mu_t^p \mathrm{d}t + \sigma_t^p \mathrm{d}W_t,$$

and use the Euler equation

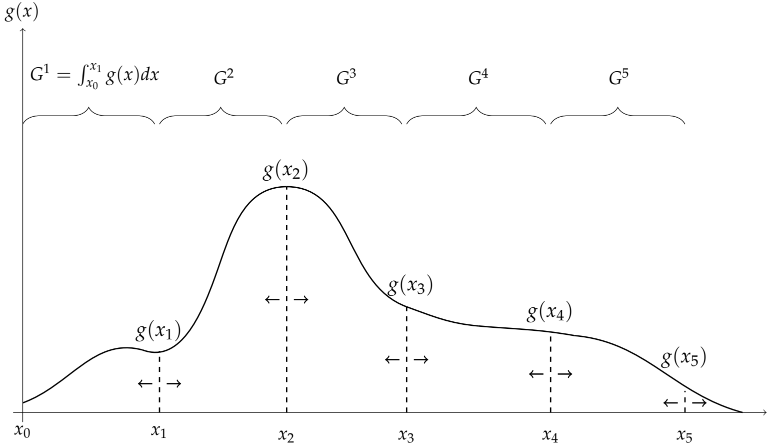
$$\frac{D\left(X_t\right)}{p\left(t, X_t\right)} + \mu_t^p + \mu^m\left(X_t\right) = -\sigma^m\left(X_t\right)\sigma_t^p$$

to derive the BSDE

Idea of Finite Volume Method

To capture the transition of the conditional distribution,

- discretize the state space into a finite number of cells or intervals
- focus on dynamics of agents on the boundaries: $x_1, \cdots, x_5$.



Kolmogorov Forward Equation

Given the density of conditional distribution $g(\cdot, t)$, X_t follows

$$\mathrm{d}X_t = \mu\left(X_t, X_t^0, g\left(\cdot, t\right)\right)\mathrm{d}t + \sigma\left(X_t, X_t^0, g\left(\cdot, t\right)\right)\mathrm{d}W_t + \sigma^0\left(X_t, X_t^0, g\left(\cdot, t\right)\right)\mathrm{d}W_t^0$$

The stochastic Kolmogorov Forward Equation is

$$\begin{aligned}\mathrm{d}g\left(x,t\right) = &-\frac{\partial}{\partial x}\left(\mu\left(x, X_t^0, g\left(\cdot,t\right)\right)g\left(x,t\right)\right)\mathrm{d}t - \frac{\partial}{\partial x}\left(\left(\sigma^0\left(x, X_t^0, g\left(\cdot,t\right)\right)\mathrm{d}W_t^0\right)g\left(x,t\right)\right)\\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\left[\left(\sigma\left(x, X_t^0, g\left(\cdot,t\right)\right)\right)^2 + \left(\sigma^0\left(x, X_t^0, g\left(\cdot,t\right)\right)\right)^2\right]g\left(x,t\right)\right\}\mathrm{d}t.\end{aligned}$$

Finite Volume Method

Discretize the space domain of $g(\cdot,t)$ into $(x_0,x_1),\cdots,(x_{N-1},x_N)$

Approximate $g(\cdot,t)$ with a finite number of

$$G_t^i = \int_{x_{i-1}}^{x_i} g(x,t)\,\mathrm{d}t.$$

Taking integration on both sides of the KFE over (x_{i-1},x_i),

$$\begin{aligned}
\mathrm{d}G_t^i = &-\left(\mu\left(x_i, X_t^0, g(\cdot,t)\right) g(x_i,t) - \mu\left(x_{i-1}, X_t^0, g(\cdot,t)\right) g(x_{i-1},t)\right)\mathrm{d}t \\
&-\left(\sigma^0\left(x_i, X_t^0, g(\cdot,t)\right) g(x_i,t) - \sigma^0\left(x_{i-1}, X_t^0, g(\cdot,t)\right) g(x_{i-1},t)\right)\mathrm{d}W_t^0 \\
&+\frac{1}{2}\frac{\partial}{\partial x}\left\{\left[\left(\sigma\left(x_i, X_t^0, g(\cdot,t)\right)\right)^2 + \left(\sigma^0\left(x_i, X_t^0, g(\cdot,t)\right)\right)^2\right] g(x_i,t)\right\}\mathrm{d}t \\
&-\frac{1}{2}\frac{\partial}{\partial x}\left\{\left[\left(\sigma\left(x_{i-1}, X_t^0, g(\cdot,t)\right)\right)^2 + \left(\sigma^0\left(x_{i-1}, X_t^0, g(\cdot,t)\right)\right)^2\right] g(x_{i-1},t)\right\}\mathrm{d}t
\end{aligned}$$

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A J-Country Macro-Finance Model

- Each country has experts and savers.
  Both have log utility with time discount factor ρ.
- Only experts in country i can hold physical assets K_t^i
  residing in the country and
  produce homogeneous consumption goods.
- Assets in each country subject to independent Brownian shocks
  exogenous vol σ
- No outside equity financing $\rightarrow$ leverage and
  financial amplification
- Perfect international trade and risk-free debt markets

Experts

Asset price q_t^i follows

$$\text{BSDE:} \quad \frac{\mathrm{d}q_t^i}{q_t^i} = \mu_t^{q,i}\mathrm{d}t + \sum_{j=1}^{J} \sigma_t^{q,i,j}\mathrm{d}Z_t^j,$$

W_t^i denote the total wealth of experts in country i,

$$\frac{\mathrm{d}W_t^i}{W_t^i} = \left(r_t + \varphi_t^i\left(R_t^i - r_t\right) - \rho\right)\mathrm{d}t + \sum_{j=1}^{J} \varphi_t^i\left(\mathbf{1}\{j=i\}\sigma + \sigma_t^{q,i,j}\right)\mathrm{d}Z_t^j$$

$$R_t^i \equiv \frac{a\psi - 1}{\psi q_t^i} + \frac{1}{\psi}\ln\left(q_t^i\right) + \psi - \delta + \mu_t^{q,i} + \sigma\sigma_t^{q,i,i}$$

Euler equation

$$R_t^i - r_t = \varphi_t^i \sum_{j=1}^{J}\left(\mathbf{1}\{j=i\}\sigma + \sigma_t^{q,i,j}\right)^2$$

Markov Equilibrium

State variables:

Experts' wealth share within a country

$$\omega_t^i = \frac{W_t^i}{q_t^i K_t^i};$$

Country's asset value share in the world

$$\zeta_t^i = \frac{q_t^i K_t^i}{\sum_{i=1}^J q_t^i K_t^i}.$$

Under symmetric states, consumption good market clearing implies

$$q_t^i = \frac{a\psi + 1}{\rho\psi + 1}, i = 1, \cdots, J$$

Numerical Experiment

Table: $\omega^s = 0.4, \zeta^s = 0.2$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
q^i	1.3045	1.3048	1.3046	1.3045	1.3034
$\sigma^{q,i,1}$	0.00079	−0.00021	−0.00023	−0.00023	−0.00013
$\sigma^{q,i,2}$	−0.00014	0.00086	−0.00014	−0.00015	−0.00045
$\sigma^{q,i,3}$	−0.00017	−0.00016	0.00083	−0.00015	−0.00035
$\sigma^{q,i,4}$	−0.00022	−0.00022	−0.00023	0.00073	−0.00006
$\sigma^{q,i,5}$	−0.00013	−0.00014	−0.00013	−0.00012	0.00053

Numerical Experiment

Table: $\omega^s = 0.5, \zeta^s = 0.2$

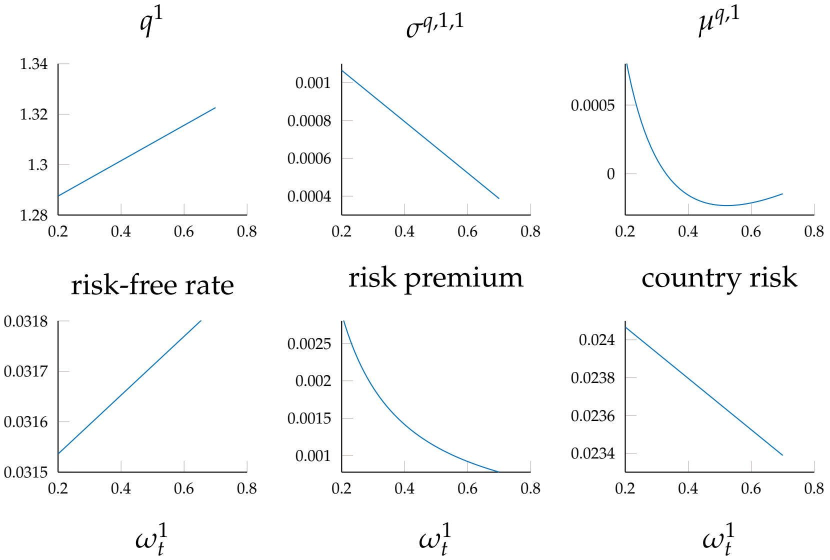
	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
q^i	1.3056	1.3059	1.3053	1.3052	1.2998
$\sigma^{q,i,1}$	0.00066	−0.00018	−0.00019	−0.00018	−0.00014
$\sigma^{q,i,2}$	−0.00011	0.00072	−0.00011	−0.00012	−0.00042
$\sigma^{q,i,3}$	−0.00014	−0.00013	0.00069	−0.00012	−0.00031
$\sigma^{q,i,4}$	−0.00018	−0.00019	−0.00019	0.00060	−0.00005
$\sigma^{q,i,5}$	−0.00011	−0.00012	−0.00012	−0.00010	0.00052

Numerical Experiment

Table: $\omega^s = 0.6, \zeta^s = 0.2$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
q^i	1.3067	1.3070	1.3060	1.3058	1.2962
$\sigma^{q,i,1}$	0.00053	−0.00014	−0.00015	−0.00014	−0.00014
$\sigma^{q,i,2}$	−0.00008	0.00058	−0.00008	−0.00008	−0.00039
$\sigma^{q,i,3}$	−0.00011	−0.00011	0.00055	−0.00009	−0.00027
$\sigma^{q,i,4}$	−0.00015	−0.00015	−0.00015	0.00047	−0.00005
$\sigma^{q,i,5}$	−0.00009	−0.00010	−0.00010	−0.00008	0.00051

Numerical Experiment



Plan

- Probabilistic approach: an oversimplified illustration
- Algorithm
- How to formulate Forward-Backward SDEs
  - recursive utility
  - dynamic optimization
  - homothetic preference with linear budget constraint
  - asset pricing
  - finite volume method
- Multiple-country macro-finance example
- Heterogeneous-agent macro example

Heterogeneous-Agent Model

A model considered by Fernandez-Villaverde, Hurtado and Nuno.

Two types of agents

- a representative expert: log utility with time discount factor ρ holds physical assets K_t, and earns rent r_t^k
- a continuum of household:

$$E_0 \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} \mathrm{d}t \right]$$

  hold risk-free debt with credit constraint $a_t \geq 0$, and labor productivity follows two-state Markov chain $\{z_1, z_2\}$

$$\begin{bmatrix} 1-\lambda_1\Delta & \lambda_1\Delta \\ \lambda_2\Delta & 1-\lambda_2\Delta \end{bmatrix}$$

Production

Production function: $Y_t = K_t^{\alpha} L_t^{1-\alpha}$

Labor wage: $w_t = (1-\alpha)\frac{Y_t}{L_t}$

Rent of physical assets: $r_t^k = \alpha \frac{Y_t}{K_t}$

Law of motion for physical assets:

$$\frac{\mathrm{d}K_t}{K_t} = (\iota_t - \delta)\,\mathrm{d}t + \sigma \mathrm{d}W_t,$$

W_t is the aggregate shock.

the market clearing condition of the final goods

$$Y_t = C_t^e + C_t^h + \iota_t K_t,$$

Expert's Dynamic Optimization

Expert's net worth N_t is

$$\frac{\mathrm{d}N_t}{N_t} = \left(r_t + x_t \left(r_t^k - \delta - r_t \right) - \frac{c_t}{N_t} \right) \mathrm{d}t + \sigma x_t \mathrm{d}W_t,$$

Optimality conditions are

$$\begin{aligned} c_t &= \rho N_t \\ r_t^k - \delta - r_t &= x_t \sigma^2 \end{aligned}$$

Households' Dynamic Optimization I

The wealth of a household follows

$$\mathrm{d}a_t = \underbrace{(w_t z_t + r_t a_t - c_t)}_{\equiv s(a_t, z_t)} \mathrm{d}t$$

A household's expected life-time utility V_t^i follows the BSDE

$$\mathrm{d}V_t^i = -\left(\frac{c_t^{1-\gamma} - 1}{1-\gamma} - \rho V_t^i\right) \mathrm{d}t + U_t^{v,i} \mathrm{d}\Lambda_t^i + Z_t^{v,i} \mathrm{d}W_t,$$

where Λ_t^i is the Poisson shock at state i.

The solution of the BSDE is $\left(V_t^i, U_t^{v,i}, Z_t^{v,i}, i = 1, 2\right)$.

Households' Dynamic Optimization II

The co-state variable Y_t^i follows the BSDE

$$\mathrm{d}Y_t^i = -\left(r_t Y_t^i - \rho Y_t^i\right)\mathrm{d}t + U_t^{y,i}\mathrm{d}\Lambda_t^i + Z_t^{y,i}\mathrm{d}W_t$$

with the solution $\left(Y_t^i, U_t^{y,i}, Z_t^{y,i}, i = 1, 2\right)$. The Hamiltonian is

$$H^i(a, c, y) = \frac{c^{1-\gamma} - 1}{1-\gamma} + (w_t z_i + r_t a - c)\, y$$

when $a_t > 0$. The optimal consumption c_t^i satisfies $\left(c_t^i\right)^{-\gamma} = Y_t^i$.

the relationship between Y_t^i and the value function $V^i(a_t, \cdot)$

$$Y_t^i = \frac{\partial V^i}{\partial a}(a_t, \cdot).$$

Stochastic KFE and FVM

The aggregate state variables are N_t and the density $g\left(t,a,z\right)$.

The stochastic KFE is

$$\frac{\partial g\left(t,a,z_i\right)}{\partial t}=-\frac{\partial}{\partial a}\left(\left(w_t z_i+r_t a_t-c_t\right)g\left(t,a,z_i\right)\right)-\lambda_i g\left(t,a,z_i\right)+\lambda_j g\left(t,a,z_j\right),i\neq j,$$

Given $0=x_0<x_1<\cdots<x_N$, discretize the space into N intervals.

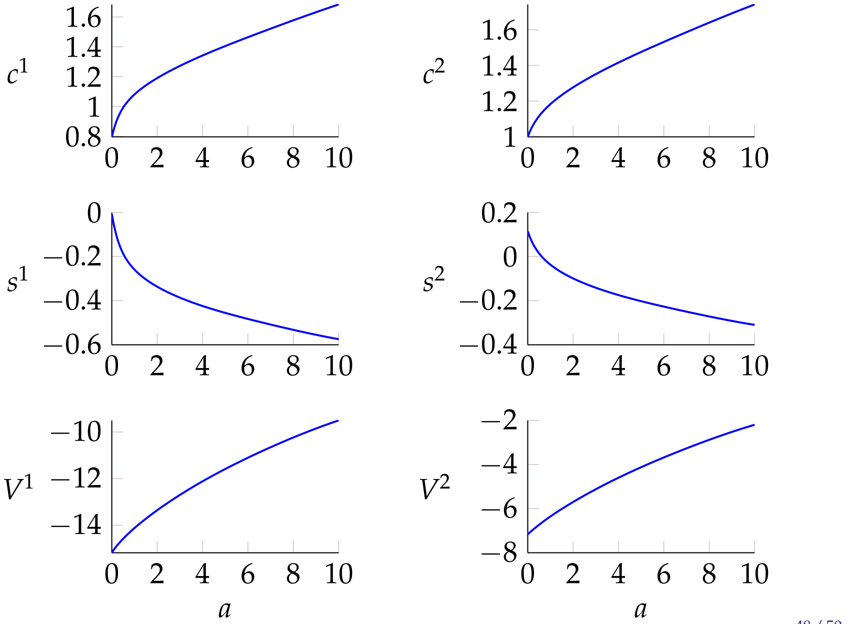
Approximate $G\left(t,a,z_i\right)$ with $G^n(t,z_i),n=1,\cdots,N$

$$G^n\left(t,z_i\right)=\int_{x_{n-1}}^{x_n}\mathrm{d}G\left(t,a,z_i\right)=\int_{x_{n-1}}^{x_n}g\left(t,a,z_i\right)\mathrm{d}a.$$

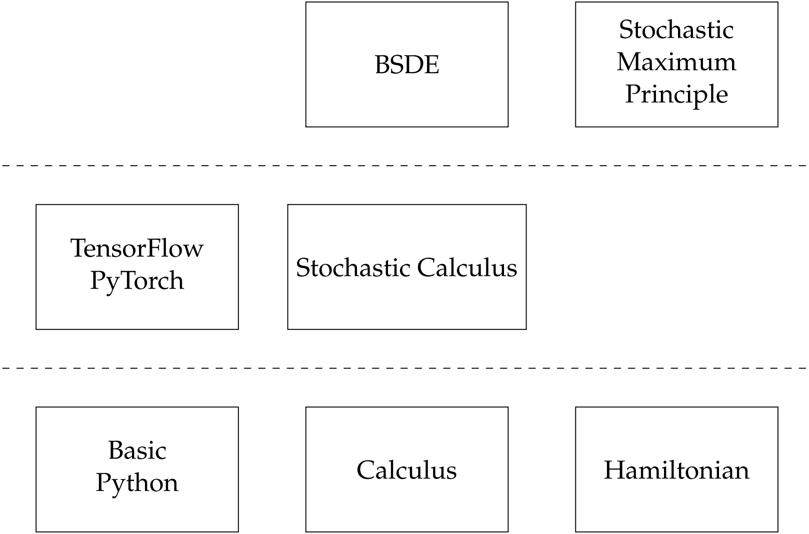
$G^n\left(t,z_i\right)$ follows

$$\frac{\partial G^n\left(t,z_i\right)}{\partial t}=-\left(s\left(x_n,z_i\right)g\left(t,x_n,z_i\right)-s\left(x_{n-1},z_i\right)g\left(t,x_{n-1},z_i\right)\right)-\lambda_i G^n\left(t,z_i\right)+\lambda_j G^n\left(t,z_j\right).$$

Numerical Experiment



Hurdles


BSDE
Stochastic Maximum Principle
TensorFlow PyTorch
Stochastic Calculus
Basic Python
Calculus
Hamiltonian

Final Remarks

Let's build large models in economics !

Binding constraints:

- high-performance GPUs
- manpower